

General monogamy relation of multi-qubit systems in terms of squared Rényi- α entanglement

Wei Song^{1,*}, Yan-Kui Bai^{2,†}, Mou Yang³, Ming Yang⁴, and Zhuo-Liang Cao¹

¹ *Institute for Quantum Control and Quantum Information, and School of Electronic and Information Engineering, Hefei Normal University, Hefei 230601, China*

² *College of Physics Science and Information Engineering and Hebei Advance Thin Films Laboratory, Hebei Normal University, Shijiazhuang, Hebei 050024, China*

³ *Laboratory of Quantum Engineering and Quantum Materials, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou 510006, China*

⁴ *School of Physics and Material Science, Anhui University, Hefei 230601, China*

We prove that the squared Rényi- α entanglement (SR α E), which is the generalization of entanglement of formation (EOF), obeys a general monogamy inequality in an arbitrary N -qubit mixed state. Furthermore, for a class of Rényi- α entanglement, we prove that the monogamy relations of the SR α E have a hierarchical structure when the N -qubit system is divided into k parties. As a byproduct, the analytical relation between the Rényi- α entanglement and the squared concurrence is derived for bipartite $2 \otimes d$ systems. Based on the monogamy properties of SR α E, we can construct the corresponding multipartite entanglement indicators which still work well even when the indicators based on the squared concurrence and EOF lose their efficacy. In addition, the monogamy property of the μ -th power of Rényi- α entanglement is analyzed.

PACS numbers: 03.67.Mn, 03.65.Ud, 03.65.Yz

I. INTRODUCTION

Monogamy of entanglement (MoE) [1] is an essential feature in many-body quantum systems, which means that quantum entanglement cannot be shared freely in multipartite systems [2]. Coffman, Kundu, and Wootters established the first quantitative characterization of the MoE for the squared concurrence (SC) [3] in an arbitrary three-qubit quantum state [4]. Furthermore, Osborne and Verstraete generalized this monogamy relation to the N -qubit case [5],

$$C^2(\rho_{A_1|A_2\dots A_n}) - C^2(\rho_{A_1A_2}) - \dots - C^2(\rho_{A_1A_n}) \geq 0. \quad (1)$$

where $C^2(\rho_{A_1|A_2\dots A_n})$ quantifies bipartite entanglement in the partition $A_1|A_2\dots A_n$ and $C^2(\rho_{A_1A_i})$ characterizes two-qubit entanglement with $i = 2, 3, \dots, n$. The MoE of SC can be used to characterize the entanglement structure in multipartite quantum systems and detect the existence of multi-qubit entanglement in dynamical procedures [6–16]. Moreover, there are also many works devoted to the topic of entanglement monogamy [17–24] and similar monogamy relations were also established for Gaussian systems [25–27], squashed entanglement [28–30], entanglement negativity [31–35] and Rényi- α entanglement [36, 37].

A genuine three-qubit entanglement measure named “three-tangle” was obtained from the MoE of SC in three-qubit pure states [4]. However, there exists a kind of three-qubit mixed states which is entangled but without two-qubit concurrence and three-tangle [38], and the similar case also exists in N -qubit systems [39]. Recently, it was indicated that the squared entanglement of formation (SEF) [3] obeys the monogamy relation in multiqubit systems [40–45]. In particular, it was

proved analytically that the SEF is monogamous in an arbitrary N -qubit mixed state [42],

$$E_f^2(\rho_{A_1|A_2\dots A_n}) - E_f^2(\rho_{A_1A_2}) - \dots - E_f^2(\rho_{A_1A_n}) \geq 0, \quad (2)$$

which overcomes the flaw of the MoE of SC and can be utilized to detect all multiqubit entanglement. Rényi- α entanglement (R α E) [46] is also well-defined entanglement measure which is the generalization of entanglement of formation (EOF) and has the merits for characterizing quantum phases with differing computational power [47], ground state properties in many-body systems [48], and topologically ordered states [49–52]. Therefore, it is natural to study the MoE of the R α E and its applications in multipartite entanglement detection. Kim and Sanders proved that the R α E with the order $\alpha \geq 2$ obeys a monogamy inequality in N -qubit systems [36], but this monogamy relation does not cover the case of EOF which corresponds to the R α E with the order $\alpha = 1$. Whether or not there exists a general monogamy relation via the R α E is yet to be resolved.

In this paper, we analyze the properties of the squared Rényi- α entanglement (SR α E) and prove that the SR α E with the order $\alpha \geq (\sqrt{7} - 1)/2 \simeq 0.823$ obeys a general monogamy relation in an arbitrary N -qubit mixed state. This result provides a broad class of new monogamy inequalities including the monogamy relation of the SEF in Eq. (2) as a special case. Furthermore, it is proved that the monogamy relations of SR α E have a hierarchical structure when the N qubit systems is divided into k parties. As a byproduct, we give an analytical expression of the R α E as a function of SC in $2 \otimes d$ systems. The monogamy relations of the SR α E can be utilized to detect the multipartite entanglement and the SR α E-based indicators we construct can work well even when the corresponding ones based on the SC and SEF lose their efficacy. Finally, we analyze the monogamy property of the μ -th power of Rényi- α entanglement.

*Electronic address: weis@hfnu.edu.cn

†Electronic address: ykbai@semi.ac.cn

II. MONOGAMY INEQUALITIES FOR SR α E IN N -QUBIT SYSTEMS

For a bipartite pure state $|\psi\rangle_{AB}$, the R α E is defined as [46]

$$E_\alpha(|\psi\rangle_{AB}) := S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2(\text{tr} \rho_A^\alpha) \quad (3)$$

where the Rényi- α entropy is $S_\alpha(\rho_A) = [\log_2(\sum_i \lambda_i^\alpha)]/(1-\alpha)$ with α being a nonnegative real number and λ_i being the eigenvalue of reduced density matrix ρ_A . The Rényi- α entropy $S_\alpha(\rho)$ converges to the von Neumann entropy when the order α tends to 1. For a bipartite mixed state ρ_{AB} , the R α E is defined via the convex-roof extension

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \quad (4)$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. In particular, for a two-qubit mixed state, the R α E with $\alpha \geq 1$ has an analytical formula which is expressed as a function of the SC [36]

$$E_\alpha(\rho_{AB}) = f_\alpha[C^2(\rho_{AB})] \quad (5)$$

where the function $f_\alpha(x)$ has the form

$$f_\alpha(x) = \frac{1}{1-\alpha} \log_2 \left[\left(\frac{1-\sqrt{1-x}}{2} \right)^\alpha + \left(\frac{1+\sqrt{1-x}}{2} \right)^\alpha \right]. \quad (6)$$

Recently, Wang *et al* further proved that the formula in Eq. (5) holds for the order $\alpha \geq (\sqrt{7}-1)/2 \simeq 0.823$ [53].

Before presenting the main results of this paper, we first give three lemmas as follows.

Lemma 1. The squared Rényi- α entanglement $E_\alpha^2(C^2)$ with $\alpha \geq (\sqrt{7}-1)/2$ in two-qubit mixed states varies monotonically as a function of the squared concurrence C^2 .

Proof: This lemma holds if the first-order derivative $\partial E_\alpha^2 / \partial x > 0$ with $x = C^2$. After a direct calculation, we have

$$\frac{\partial E_\alpha^2}{\partial x} = \frac{\alpha (B^{\alpha-1} - A^{\alpha-1}) \ln [2^{-\alpha} (A^\alpha + B^\alpha)]}{(1-\alpha)^2 (A^\alpha + B^\alpha) \sqrt{1-x} (\ln 2)^2} \quad (7)$$

which is always nonnegative for $0 \leq x \leq 1$ and $\alpha \geq 0$ with the parameters $A = 1 + \sqrt{1-x}$ and $B = 1 - \sqrt{1-x}$, and the equality holds only at the boundary of x . Thus we obtain that E_α^2 is monotonically increasing as a function of the squared concurrence, which completes the proof.

Lemma 2. The squared Rényi- α entanglement $E_\alpha^2(C^2)$ with $\alpha \geq (\sqrt{7}-1)/2$ is convex as a function of the squared concurrence C^2 .

Lemma 3. The Rényi- α entanglement $E_\alpha(C^2)$ with $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ is monotonic and concave as a function of the squared concurrence C^2 .

The proofs for lemma 2 and lemma 3 can be seen from Appendices A and B.

Now, we give the main results of this paper.

Theorem 1. For an arbitrary N -qubit mixed state $\rho_{A_1 A_2 \dots A_n}$, the squared Rényi- α entanglement satisfies the monogamy relation

$$E_\alpha^2(\rho_{A_1|A_2 \dots A_n}) - E_\alpha^2(\rho_{A_1 A_2}) - \dots - E_\alpha^2(\rho_{A_1 A_n}) \geq 0, \quad (8)$$

where $E_\alpha^2(\rho_{A_1|A_2 \dots A_n})$ quantifies the entanglement in the partition $A_1|A_2 \dots A_n$ and $E_\alpha^2(\rho_{A_1 A_i})$ quantifies the one in two-qubit subsystem $A_1 A_i$ with the order $\alpha \geq (\sqrt{7}-1)/2$.

Proof. We first consider the monogamy relation in an N -qubit pure state $|\psi\rangle_{A_1 A_2 \dots A_n}$. The entanglement $E_\alpha(|\psi\rangle_{A_1|A_2 \dots A_n})$ can be evaluated using Eq.(5) since the subsystem $A_2 \dots A_n$ can be regarded as a logic qubit. Thus, we can obtain

$$\begin{aligned} E_\alpha^2(|\psi\rangle_{A_1|A_2 \dots A_n}) &= E_\alpha^2[C_{A_1|A_2 \dots A_n}^2(|\psi\rangle)] \\ &\geq E_\alpha^2\left(\sum_{i=2}^n C_{A_1 A_i}^2\right) \\ &\geq \sum_{i=2}^n E_\alpha^2(\rho_{A_1 A_i}), \end{aligned} \quad (9)$$

where in the first inequality we have used the monogamy relation of squared concurrence $C_{A_1|A_2 \dots A_n}^2 \geq \sum_{i=2}^n C_{A_1 A_i}^2$ [4, 5] and the monotonically increasing property of $E_\alpha^2(C^2)$ (lemma 1), and in the second inequality we have further used the convex property of $E_\alpha^2(C^2)$ (lemma 2).

Next, we analyze the monogamy relation in an N -qubit mixed state $\rho_{A_1 A_2 \dots A_n}$. In this case, the formula of Rényi- α entanglement in Eq.(5) cannot be applied to $E_\alpha(\rho_{A_1|A_2 \dots A_n})$ since the subsystem $A_2 \dots A_n$ is not a logic qubit in general. But we can still use the formula in Eq.(4) which comes from the convex roof extension of the pure state entanglement. Therefore, we have

$$E_\alpha(\rho_{A_1|A_2 \dots A_n}) = \min \sum p_i E_\alpha(|\psi_i\rangle_{A_1|A_2 \dots A_n}), \quad (10)$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of the mixed state $\rho_{A_1 A_2 \dots A_n}$. Assuming that the optimal decomposition for Eq.(10) is

$\rho_{A_1 A_2 \dots A_n} = \sum_{j=1}^m p_j |\psi_j\rangle_{A_1 A_2 \dots A_n} \langle \psi_j|$, we have

$$\begin{aligned}
E_\alpha^2(\rho_{A_1 A_2 \dots A_n}) &= \left[\sum_j p_j E_\alpha(|\psi_j\rangle_{A_1 A_2 \dots A_n}) \right]^2 \\
&= \left\{ \sum_j p_j E_\alpha[C_{A_1 A_2 \dots A_n}(|\psi_j\rangle)] \right\}^2 \\
&\geq \left\{ E_\alpha \left[\sum_j p_j C_{A_1 A_2 \dots A_n}(|\psi_j\rangle) \right] \right\}^2 \\
&\geq \{E_\alpha[C_{A_1 A_2 \dots A_n}(\rho)]\}^2 \\
&= E_\alpha^2[C_{A_1 A_2 \dots A_n}^2(\rho)] \\
&\geq E_\alpha^2 \left[\sum_{i=2}^n C^2(\rho_{A_1 A_i}) \right] \\
&\geq \sum_{i=2}^n E_\alpha^2[C^2(\rho_{A_1 A_i})] \\
&= \sum_{i=2}^n E_\alpha^2(\rho_{A_1 A_i}), \tag{11}
\end{aligned}$$

where we have used in the second equality the pure state formula of the R α E and taken the $E_\alpha(C)$ as a function of the concurrence C for $\alpha \geq (\sqrt{7} - 1)/2$; in the third inequality we have used the monotonically increasing and convex properties of $E_\alpha(C)$ as a function of the concurrence [36]; in the fourth inequality we have used the convex property of concurrence for mixed states; and in the sixth and seventh inequalities we use the monotonically increasing and convex properties of $E_\alpha^2(C^2)$ as a function of the squared concurrence (lemmas 1 and 2). Thus we have completed the proof of theorem 1.

Theorem 2. For a bipartite $2 \otimes d$ mixed state ρ_{AC} , the Rényi- α entanglement has an analytical expression

$$E_\alpha(\rho_{AC}) = f_\alpha[C^2(\rho_{AC})] \tag{12}$$

where the order α ranges in the region $[(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$.

Proof. Suppose that the optimal decomposition for $E_\alpha(\rho_{AC})$ is $\{p_i, |\psi_i\rangle_{AC}\}$, we have

$$\begin{aligned}
E_\alpha(\rho_{AC}) &= \sum_i p_i E_\alpha(|\psi_i\rangle_{AC}) \\
&= \sum_i p_i E_\alpha[C^2(|\psi_i\rangle_{AC})] \\
&\leq \sum_i q_i E_\alpha[C^2(|\varphi^i\rangle_{AC})] \\
&\leq E_\alpha \left[\sum_i q_i C^2(|\varphi^i\rangle_{AC}) \right] \\
&= E_\alpha[C^2(\rho_{AC})] \tag{13}
\end{aligned}$$

where in the third inequality we have assumed that $\{q_i, |\varphi^i\rangle_{AC}\}$ is the optimal decomposition for the squared concurrence $C^2(\rho_{AC})$, and in the fourth inequality we have used the property that E_α is a concave function of the squared

concurrence C^2 for the order $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ (lemma 3).

On the other hand, we can derive

$$\begin{aligned}
E_\alpha(\rho_{AC}) &= \sum_i p_i E_\alpha(|\psi^i\rangle_{AC}) \\
&= \sum_i p_i E_\alpha[C(|\psi^i\rangle_{AC})] \\
&\geq E_\alpha \left[\sum_i p_i C(|\psi^i\rangle_{AC}) \right] \\
&\geq E_\alpha \left[\sum_k r_k C(|\phi^k\rangle_{AC}) \right] \\
&= E_\alpha[C(\rho_{AC})] \tag{14}
\end{aligned}$$

where the third inequality holds due to the convex property of $E_\alpha(C)$ as a function of concurrence C for $\alpha \geq (\sqrt{7} - 1)/2$ [36, 53], and the fourth inequality is satisfied due to $\{r_k, |\phi^k\rangle_{AC}\}$ being the optimal pure-state decomposition for $C(\rho_{AC})$.

Combining Eq. (13) with Eq. (14), we have $E_\alpha[C(\rho_{AC})] \leq E_\alpha[\rho_{AC}] \leq E_\alpha[C^2(\rho_{AC})]$. Due to $E_\alpha[C(\rho_{AC})]$ and $E_\alpha[C^2(\rho_{AC})]$ being the same expression, we obtain the equality shown in Eq. (12) and the proof is completed.

Corollary 1. For the order $\alpha > (\sqrt{13} - 1)/2 \simeq 1.303$, the Rényi- α entanglement in bipartite $2 \otimes d$ systems obeys the following relation

$$E_\alpha(\rho_{AC}) \geq f_\alpha[C^2(\rho_{AC})], \tag{15}$$

which provides a nontrivial lower bound for the entanglement.

The proof of this corollary is straightforward according to Eq. (14).

Theorem 3. For an arbitrary N -qubit mixed state $\rho_{A_1 A_2 \dots A_N}$, there exist a set of k -partite hierarchical monogamy relations

$$E_\alpha^2(\rho_{A_1 A_2 \dots A_N}) \geq \sum_{i=2}^{k-1} E_\alpha^2(\rho_{A_1 A_i}) + E_\alpha^2(\rho_{A_1 A_{k+1} \dots A_N}) \tag{16}$$

where the number of parties is $k = \{3, 4, \dots, N\}$ and the order α in the region $[(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$.

Proof. We first consider a tripartite $2 \otimes 2 \otimes 2^{N-2}$ mixed state ρ_{ABC} , for which we can derive

$$\begin{aligned}
&E_\alpha^2(\rho_{A|BC}) - E_\alpha^2(\rho_{AB}) - E_\alpha^2(\rho_{AC}) \\
&= f_\alpha^2[C^2(\rho_{A|BC})] - f_\alpha^2[C^2(\rho_{AB})] - f_\alpha^2[C^2(\rho_{AC})] \\
&\geq f_\alpha^2[C^2(\rho_{AB}) + C^2(\rho_{AC})] \\
&\quad - f_\alpha^2[C^2(\rho_{AB})] - f_\alpha^2[C^2(\rho_{AC})] \\
&\geq 0 \tag{17}
\end{aligned}$$

where in the first equality we have used formula (12) in theorem 2, in the second inequality we have utilized the monotonic property of $f_\alpha^2(C^2)$ and the monogamy relation of the SC

$C^2(\rho_{A|BC}) \geq C^2(\rho_{AB}) + C^2(\rho_{AC})$ [5], and in the third inequality we used the convex property of $f_\alpha^2(C^2)$ in lemma 2.

After further cutting the subsystem C into a qubit C_1 and a qudit C_2 with $d = 2^{N-3}$ and applying Eq. (17) to the tripartite quantum state $\rho_{AC_1C_2}$, we can obtain

$$E_\alpha^2(\rho_{A|BC}) - E_\alpha^2(\rho_{AB}) - E_\alpha^2(\rho_{AC_1}) - E_\alpha^2(\rho_{AC_2}) \geq 0. \quad (18)$$

By the successive cut for the last party and application of the tripartite monogamy inequality, we can derive a set of hierarchical k -partite monogamy relations with $k = \{3, 4, \dots, N\}$ as shown in Eq. (16), and such that we prove theorem 3.

III. MULTIPARTITE ENTANGLEMENT INDICATORS BASED ON THE SR α E AND ITS APPLICATIONS

According to the established monogamy relations based on the R α E in Eqs. (8) and (16), we can construct two kinds of multipartite entanglement indicators

$$\tau_{\alpha(N)}^{(1)}(\rho) = \min \sum_i p_i \tau_{\alpha(N)} \left(\left| \psi_{A_1|A_2 \dots A_n}^i \right\rangle \right) \quad (19)$$

$$\begin{aligned} \tau_{\alpha(K)}^{(2)}(\rho) &= E_\alpha^2(\rho_{A_1|A_2 \dots A_n}) - \sum_{i=2}^{k-1} E_\alpha^2(\rho_{A_1|A_i}) \\ &\quad - E_\alpha^2(\rho_{A_1|A_k \dots A_n}), \end{aligned} \quad (20)$$

which can be utilized to detect the multipartite entanglement in an N -qubit state $\rho_{A_1 A_2 \dots A_n}$. The first indicator $\tau_{\alpha(N)}^{(1)}$ can detect the existence of multipartite entanglement in an N -partite system, which comes from the convex roof of pure state indicator $\tau_{\alpha(N)}(|\psi\rangle_{A_1|A_2 \dots A_n}) = E_\alpha^2(|\psi\rangle_{A_1|A_2 \dots A_n}) - \sum_{i=2}^n E_\alpha^2(\rho_{A_1 A_i})$ with the minimum being taken over all possible pure state decompositions. The second indicator $\tau_{\alpha(K)}^{(2)}$ can detect the multipartite entanglement in a K -partite system with $K \in \{3, 4, \dots, N\}$, which is the residual entanglement in the hierarchical monogamy inequality shown in Eq. (20). In the following, we give three examples as applications of the above entanglement indicators.

Example 1. We consider a three-qubit pure state $|\psi(p)\rangle = \sqrt{p}|GHZ_3\rangle - \sqrt{1-p}|W_3\rangle$ which is the superposition of a GHZ state and a W state with $|GHZ_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|W_3\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. The three-tangle τ is a tripartite entanglement measure based on the monogamy relation of the SC and defined as $\tau(|\psi\rangle_{ABC}) = C_{A|BC}^2 - C_{AB}^2 - C_{AC}^2$ [4]. For the quantum state $|\psi(p)\rangle$, its three-tangle is $\tau(|\psi(p)\rangle) = p^2 - 8\sqrt{6}\sqrt{p(1-p)^3}/9$ which has two zero points at $p_1 = 0$ and $p_2 \simeq 0.627$ resulting in some flaw for the entanglement detection [38, 42]. In this case, we use the newly introduced multipartite entanglement indicator $\tau_{\alpha}^{(1)}$ shown in Eq. (19). It is direct to calculate the value of $\tau_{\alpha}^{(1)}[|\psi(p)\rangle]$ since the R α E has an analytical formula for two-qubit quantum states and the convex roof extension is not needed for the pure state case. In

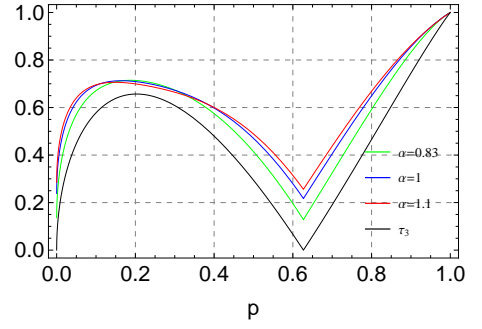


FIG. 1: (color online). The indicator $\tau_{\alpha}^{(1)}$ for the superposition state $|\psi(p)\rangle$ with $\alpha = 0.83$ (green line), $\alpha = 1$ (blue line), and $\alpha = 1.1$ (red line). As an comparison, we also plot the three-tangle of $|\psi(p)\rangle$ with black line.

Fig.1, we plot the three-tangle and the indicator $\tau_{\alpha}^{(1)}$ for the order $\alpha = 0.83, 1, 1.1$. As shown in the figure, the indicator $\tau_{\alpha}^{(1)}$ is always positive for the different order α in contrast to the three-tangle τ_3 having two zero points, which detects all the genuine tripartite entanglement in all the region $p \in [0, 1]$.

Example 2. Lohmayer *et al* found that there exists a kind of three-qubit mixed states [38],

$$\rho_{ABC} = p|GHZ_3\rangle\langle GHZ_3| + (1-p)|W_3\rangle\langle W_3| \quad (21)$$

which is entangled but without two-qubit concurrence and three-tangle for the parameter $p \in [0.292, 0.627]$. Now we use the indicator $\tau_{\alpha}^{(1)}$ to detect the genuine three-qubit entanglement in the mixed state. After some analysis, we obtain that the optimal pure state decomposition for $p \leq 0.627$ is

$$\rho_{ABC} = (F/3) \sum_{j=0}^2 |\psi^j(p_0)\rangle\langle\psi^j(p_0)| + (1-F)|W_3\rangle\langle W_3|$$

in which $F = p/p_0$ with $p_0 = 0.627$ and the component $|\psi^j(p_0)\rangle = \sqrt{p_0}|GHZ_3\rangle - e^{(2\pi i/3)j}\sqrt{1-p_0}|W_3\rangle$. Then we can derive $\tau_{\alpha}^{(1)}(\rho_{A|BC}) = F\tau_{\alpha}^{(1)}(|\psi^0(p_0)\rangle) + (1-F)\tau_{\alpha}^{(1)}(|W_3\rangle)$. In Fig.2, we plot the the indicator $\tau_{\alpha}^{(1)}(\rho_{ABC})$ as a function of parameters p and α . As shown in the figure, the values of this set of indicators are always positive, which detect the existence of the genuine three-qubit entanglement in the mixed state. It is noted that the case with the order $\alpha = 1$ coincides with the result of the SEF-based indicator [42] since the R α E converge to the EOF in this case.

Example 3. The three-tangle based on the monogamy relation of the SC cannot detect the tripartite entanglement in the W state. However, the SR α E-based indication $\tau_{\alpha}^{(2)}$ can still work in this case. We consider the N -qubit W state in the form $|W_N\rangle = 1/\sqrt{N}(|10 \dots 0\rangle + |01 \dots 0\rangle + \dots + |00 \dots 1\rangle)$. When the quantum system is divided into K parties with $K \in [3, 4, \dots, N]$, there are a set of hierarchical monogamy relations. The corresponding indicator can be written as $\tau_{\alpha(K)}(|W_N\rangle) = E_\alpha^2(C_{A_1|A_2 \dots A_N}^2) - (k-2)E_\alpha^2(C_{A_1 A_2}^2) - E_\alpha^2(C_{A_1|A_k \dots A_N}^2)$, where $C_{A_1|A_2 \dots A_N}^2 = 4(N-1)/N^2$, $C_{A_1 A_2}^2 = 4/N^2$, and $C_{A_1|A_k \dots A_N}^2 = 4(N-k+1)/N^2$. In Table I, we calculate

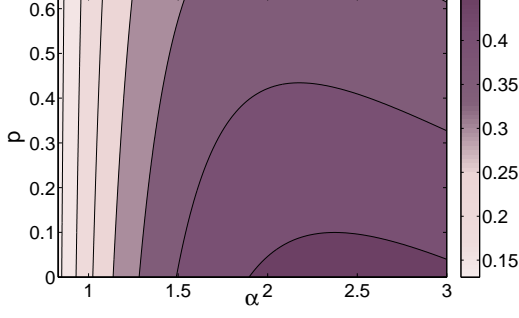


FIG. 2: (color online). The indicator $\tau_{\alpha(3)}^{(1)}$ with the order $\alpha \in (0.83, 3)$ detects the existence of the genuine three-qubit entanglement in the mixed state ρ_{ABC} when the parameter $p \leq p_0$.

	$\alpha = 0.95$	$\alpha = 1$	$\alpha = 1.05$	$\alpha = 1.10$	$\alpha = 1.15$
$k = 3$	0.0600	0.0626	0.0644	0.0656	0.0662
$k = 4$	0.1136	0.1178	0.1205	0.1219	0.1225
$k = 5$	0.1594	0.1642	0.1669	0.1680	0.1678
$k = 6$	0.1954	0.2000	0.2021	0.2023	0.2010
$k = 7$	0.2181	0.2219	0.2231	0.2222	0.2199

TABLE I: The values of the indicator $\tau_{\alpha(K)}(|W_7\rangle)$ for the different party number k and entanglement order α .

the indicator $\tau_{\alpha(K)}^{(2)}$ for a 7-qubit W state, where the party number k ranges in $[3, 7]$ and the order α is chosen as 0.95, 1, 1.05, 1.1, and 1.15. The nonzero values of this indicator reveal the multipartite entanglement in the W state.

IV. DISCUSSION AND CONCLUSION

We have considered the monogamy relations for the SR α E in multiqubit systems. However, it is still an open problem that whether this result can be extended to the multi-level systems. Ou pointed out that the SC is not monogamous in a three-qutrit quantum state $|\psi\rangle_{ABC} = (|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle)/\sqrt{6}$ [54]. When we use the monogamy relation of the SR α E, it is found that

$$E_{\alpha}^2(|\psi\rangle_{A|BC}) - E_{\alpha}^2(\rho_{AB}) - E_{\alpha}^2(\rho_{AC}) \simeq 0.51211, \quad (22)$$

which is monogamous for an arbitrary value of the order α . Next, we consider a four-partite mixed state $\rho_{A_1 A_2 A_3 A_4}$ in $2 \otimes d_2 \otimes d_3 \otimes d_4$ systems. Suppose that $C_{A_1|A_2 A_3 A_4}^2 = 0.7$, and $C_{A_1 A_2}^2 = C_{A_1 A_3}^2 = C_{A_1 A_4}^2 = 0.35$. In this case, neither the SC or the SEF is monogamous and we have

$$C^2(\rho_{A_1|A_2 A_3 A_4}) - \sum_{i=2}^4 C^2(\rho_{A_1 A_i}) = -0.35$$

$$E_f^2(\rho_{A_1|A_2 A_3 A_4}) - \sum_{i=2}^4 E_f^2(\rho_{A_1 A_i}) = -0.037. \quad (23)$$

But the monogamy relation of the SR α E still works for this mixed state and we can get

$$E_{\alpha}^2(\rho_{A_1|A_2 A_3 A_4}) - \sum_{i=2}^4 E_{\alpha}^2(\rho_{A_1 A_i}) = 0.052, \quad (24)$$

where the order $\alpha = 1.2$ has been chosen.

Beside the monogamy relations we have established in terms of the SR α E, the similar relations can also be generalized to the μ -th power of the R α E. After some derivation, we can obtain the following theorem and its proof can be seen from Appendix C.

Theorem 4. For an arbitrary three-qubit mixed state $\rho_{A_1 A_2 A_3}$, the μ -th power Rényi- α entanglement obeys the monogamy relation

$$E_{\alpha}^{\mu}(\rho_{A_1|A_2 A_3}) \geq E_{\alpha}^{\mu}(\rho_{A_1 A_2}) + E_{\alpha}^{\mu}(\rho_{A_1 A_3}) \quad (25)$$

where the order $\alpha \geq (\sqrt{7} - 1)/2 \simeq 0.823$ and the power $\mu \geq 2$. Moreover, in N -qubit systems, the following monogamy relation is also satisfied

$$E_{\alpha}^{\mu}(\rho_{A_1|A_2 \dots A_n}) \geq \sum_{i=2}^{k-1} E_{\alpha}^{\mu}(\rho_{A_1 A_i}) + E_{\alpha}^{\mu}(\rho_{A_1|A_k \dots A_n}) \quad (26)$$

where the power $\mu \geq 2$ and the order $\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$.

We have thus proved the monogamy relation of the SR α E in an arbitrary multi-qubit systems. Our results provide a broad class of new monogamy inequalities which include the previous result in terms of the SEF as a special case. Moreover, we have proved that the monogamy relations of the SR α E possess a hierarchical structure when the N -qubit system is divided into k parties. These new derived monogamy relations can be used to construct multipartite entanglement indicators in N -qubit systems, which still work well even when the corresponding ones based on the SC and SEF lose their efficacy. We also derived an analytical expression for the R α E as a function of the SC in bipartite $2 \otimes d$ systems. Finally, we analyze the monogamy property of the μ -th power of Rényi- α entanglement. It is still an open problem yet to be answered that whether there exists a monogamy relation for the SR α E in higher-dimensional systems.

Acknowledgment

The authors would like to thank Jun-Sheng Feng, and Yong-Sheng Tang for helpful discussions. This work was supported by NSF-China under Grant Nos. 11575051, 11374085, 11274010, and 11274124, the Specialized Research Fund for the Doctoral Program of Higher Education (no. 20113401110002), and the 136 Foundation of Hefei Normal University under Grant No.2014136KJB04. Y.-K. Bai was also supported by the Hebei NSF under Grant No. A2016205215.

Appendix A: Proof of lemma 2

This lemma holds if the second-order derivative $\partial^2 E_\alpha^2 / \partial x^2 > 0$ for $\alpha \geq (\sqrt{7} - 1)/2$. We first consider the squared Rényi- α entanglement for $\alpha \geq 1$. In this case, we define a function

$$h_\alpha = \frac{\partial^2 [(1 - \alpha)^2 E_\alpha^2]}{\partial^2 x^2} \quad (\text{A1})$$

on the domain $\mathcal{D} = \{(x, \alpha) | 0 \leq x \leq 1, \alpha \geq 1\}$ with x being the squared concurrence. The nonnegativity of h_α can guarantee the nonnegative $\partial^2 E_\alpha^2 / \partial x^2$ since they are equivalent up to a positive constant. After some derivation, we have

$$\begin{aligned} h_\alpha = & \Lambda \left\{ \alpha (A^{\alpha-1} - B^{\alpha-1})^2 + \left[\frac{(B^{\alpha-1} - A^{\alpha-1})(A^\alpha + B^\alpha)}{\sqrt{1-x}} \right. \right. \\ & \left. \left. + (\alpha - 1)(A^{\alpha-2} + B^{\alpha-2})(A^\alpha + B^\alpha) - \alpha (A^{\alpha-1} - B^{\alpha-1})^2 \right] \right. \\ & \left. \times \ln [2^{-\alpha} (A^\alpha + B^\alpha)] \right\} \end{aligned} \quad (\text{A2})$$

where the parameter is $\Lambda = \alpha/2(A^\alpha + B^\alpha)^2(1-x)(\ln 2)^2$ with $A = 1 + \sqrt{1-x}$ and $B = 1 - \sqrt{1-x}$. For the proof of the nonnegativity of h_α , it is sufficient to analyze its maximal or minimal value on the domain \mathcal{D} . The critical points of h_α satisfy the condition

$$\nabla h_\alpha = \left(\frac{\partial h_\alpha}{\partial x}, \frac{\partial h_\alpha}{\partial \alpha} \right) = 0. \quad (\text{A3})$$

In Fig.3 (a) and (b), we plot the solutions to equations $\partial h_\alpha / \partial x = 0$ and $\partial h_\alpha / \partial \alpha = 0$, respectively. As shown in the figure, the common solution is $\alpha = 1$ which is on the boundary of the domain \mathcal{D} . Therefore, the maximal or minimal value of h_α can arise only on the boundary of domain \mathcal{D} . Next, we consider the other two boundaries $x = 0$ and $x = 1$ on the domain \mathcal{D} of h_α . When $x = 0$, we have

$$\lim_{x \rightarrow 0} h_\alpha = \frac{\alpha^2}{8(\ln 2)^2} \quad (\text{A4})$$

which is always positive in the region $\alpha \in (1, +\infty)$. Similarly, when $x = 1$, we can derive

$$\lim_{x \rightarrow 1} h_\alpha = \frac{(1 - \alpha)^2 \alpha [3\alpha + 2(\alpha^2 + \alpha - 3) \ln 2]}{6(\ln 2)^2} \quad (\text{A5})$$

which is monotonically increasing and positive in the region $\alpha \in (1, +\infty)$. Notice that the critical points arise only on the boundary of domain \mathcal{D} , we obtain that the function h_α is non-negative in the whole range $0 \leq x \leq 1, \alpha \geq 1$ (the equality holds only at the boundary $\alpha = 1$). In Fig. 4, we plot h_α as a function of x and α , which illustrates our result. According to the equivalent relation in Eq. (A1), we have $\partial^2 E_\alpha^2 / \partial x^2 > 0$ for $\alpha > 1$. When $\alpha = 1$, E_α^2 converges to SEF and its second-order derivative is positive [42]. Therefore, the second-order derivative of E_α^2 is positive for $\alpha \geq 1$.

We further analyze the nonnegative region for the second-order derivative $\partial^2 E_\alpha^2 / \partial x^2$ when α ranges in $(0, 1)$. It is

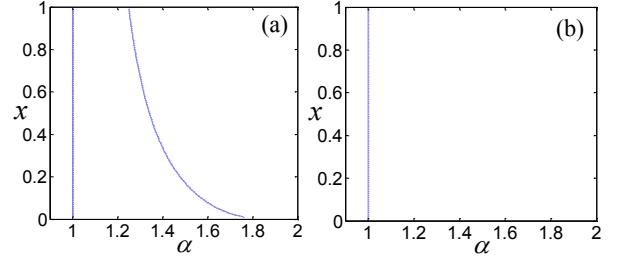


FIG. 3: (color online). The plot of the dependence of x with α which satisfies the equation (a) $\frac{\partial h_\alpha}{\partial x} = 0$ and (b) $\frac{\partial h_\alpha}{\partial \alpha} = 0$ respectively.

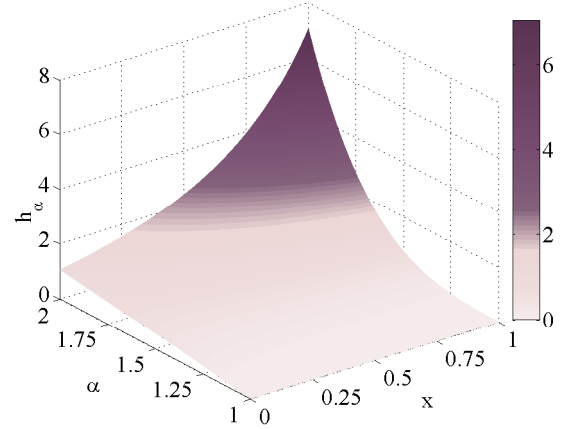


FIG. 4: (color online). The function h_α is plotted as a function of x and α for $0 \leq x \leq 1, \alpha \geq 1$, which is positive, and as a result, the SR α E is a convex function of SC.

found that, under the condition $\partial^2 E_\alpha^2 / \partial x^2 = 0$, the critical value of x increases monotonically along with the parameter α . In Fig.5(a), we plot the solution (x, α) to this critical condition, where for each fixed x there exists a value of α such that the second-order derivative of E_α^2 is zero. Due to x varying monotonically with α , we only need consider the condition $\partial^2 E_\alpha^2 / \partial x^2 = 0$ in the limit $x \rightarrow 1$. In this case, we have the derivative

$$\lim_{x \rightarrow 1} \frac{\partial^2 E_\alpha^2}{\partial x^2} = \frac{\alpha [3\alpha + 2(\alpha^2 + \alpha - 3) \ln 2]}{6(\ln 2)^2} = 0, \quad (\text{A6})$$

which gives the critical point $\alpha_{c_1} = [-(2 \ln 2 + 3) + \sqrt{(2 \ln 2 + 3)^2 + 48(\ln 2)^2}] / 4 \ln 2 \simeq 0.764$. When $\alpha \geq \alpha_{c_1}$, the second-order $\partial^2 E_\alpha^2 / \partial x^2$ is always positive. Notice that the analytical formula $E_\alpha(C^2)$ in Eq.(5) is established only for $\alpha \geq (\sqrt{7} - 1)/2 = \alpha_c$, we have $\partial^2 E_\alpha^2 / \partial x^2 > 0$ for $\alpha \in [\alpha_c, 1)$.

Combining the two positive regions $[\alpha_c, 1)$ and $[1, +\infty)$, we obtain the derivative $\partial^2 E_\alpha^2 / \partial x^2 > 0$ for $\alpha \geq (\sqrt{7} - 1)/2$, which completes the proof of lemma 2.

Appendix B: Proof of lemma 3

The R α E E_α is monotonically increasing if the first-order derivative $\partial E_\alpha / \partial x > 0$ with x being the squared concurrence. After some calculation, we have

$$\frac{\partial E_\alpha}{\partial x} = \frac{\alpha (B^{\alpha-1} - A^{\alpha-1})}{2(1-\alpha)(A^\alpha + B^\alpha)\sqrt{1-x}\ln 2} \quad (\text{B1})$$

where the parameters are $A = 1 + \sqrt{1-x}$ and $B = 1 - \sqrt{1-x}$. It is easy to verify the derivative is nonnegative in the regions $\alpha \in [0, 1)$ and $\alpha \in (1, +\infty)$. Combining the two regions with the case $\alpha = 1$ which was proved in Ref. [44], we obtain that the first-order derivative of E_α is always non-negative for $\alpha \geq 0$ and the equality holds only at the boundary of x . Therefore, the R α E is monotonically increasing.

Furthermore, the concave property of E_α as a function of C^2 holds if the second-order derivative $\partial^2 E_\alpha / \partial x^2 < 0$ with x being the squared concurrence. In order to determinate the region of α , we analyze the condition $\partial^2 E_\alpha / \partial x^2 = 0$. It is found that the value of x decreases monotonically along with the increase of α . In Fig. 5(b), we plot the relation between x and α under this condition. As shown in the figure, the critical point corresponds to the limit

$$\lim_{x \rightarrow 1} \frac{\partial^2 E_\alpha}{\partial x^2} = \frac{\alpha(\alpha^2 + \alpha - 3)}{6 \ln 2} = 0. \quad (\text{B2})$$

After some calculation, we can obtain that the critical point is $\alpha_{c2} = (\sqrt{13} - 1)/2 \simeq 1.303$. Therefore, the second-order derivative is negative when $\alpha < \alpha_{c2}$. It is noted that the analytical formula $E_\alpha(C^2)$ hold for $\alpha \geq (\sqrt{7} - 1)/2$. Thus we can get $\partial^2 E_\alpha / \partial x^2 \leq 0$ for $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ (the equality holds only at the right boundary), which results in the concave property of $E_\alpha(C^2)$. The proof of lemma 3 is completed.

Appendix C: Proof of theorem 4

According to theorem 1, we have

$$E_\alpha^2(\rho_{A_1|A_2A_3}) \geq E_\alpha^2(\rho_{A_1A_2}) + E_\alpha^2(\rho_{A_1A_3}) \quad (\text{C1})$$

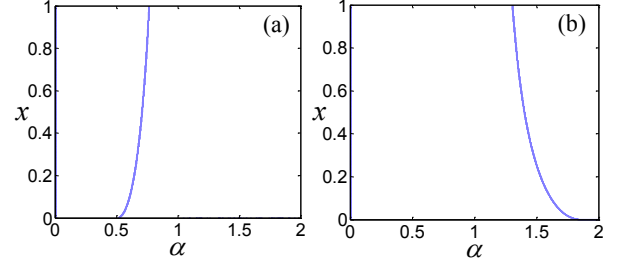


FIG. 5: (color online). The plot of the dependence of x with α using the equation (a) $\frac{\partial^2 E_\alpha^2}{\partial x^2} = 0$, (b) $\frac{\partial^2 E_\alpha}{\partial x^2} = 0$.

for $\alpha \geq (\sqrt{7} - 1)/2$. Without loss of generality, we assume the two-qubit entanglement $E_\alpha(\rho_{A_1A_2}) \geq E_\alpha(\rho_{A_1A_3})$. Then we can get

$$\begin{aligned} E_\alpha^\mu(\rho_{A_1|A_2A_3}) &\geq (E_\alpha^2(\rho_{A_1A_2}) + E_\alpha^2(\rho_{A_1A_3}))^{\frac{\mu}{2}} \\ &= E_\alpha^\mu(\rho_{A_1A_2}) \left(1 + \frac{E_\alpha^2(\rho_{A_1A_3})}{E_\alpha^2(\rho_{A_1A_2})}\right)^{\frac{\mu}{2}} \\ &\geq E_\alpha^\mu(\rho_{A_1A_2}) \left(1 + \left(\frac{E_\alpha^2(\rho_{A_1A_3})}{E_\alpha^2(\rho_{A_1A_2})}\right)^{\frac{\mu}{2}}\right) \\ &= E_\alpha^\mu(\rho_{A_1A_2}) + E_\alpha^\mu(\rho_{A_1A_3}) \end{aligned} \quad (\text{C2})$$

where in the third inequality we have used the property $(1+x)^t \geq 1+x^t$, for $x \leq 1, t \geq 1$.

For the monogamy relation of μ -th power in N -qubit systems, we first consider a tripartite mixed state in $2 \otimes 2 \otimes 2^{N-2}$ systems. According to theorem 3, the tripartite monogamy relation for the squared R α E is satisfied. Then, using the same technique in Eq. (C1), we can obtain the μ -th power monogamy inequality. Furthermore, by the successive cut for the last party and application of the tripartite monogamy relation, we can derive the k -partite inequality as shown in Eq. (26) of the main text.

-
- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
 - [2] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
 - [3] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
 - [4] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
 - [5] T. J. Osborne, and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
 - [6] Y.-K. Bai, D. Yang, and Z. D. Wang, Phys. Rev. A **76**, 022336 (2007).
 - [7] Y.-K. Bai, and Z. D. Wang, Phys. Rev. A **77**, 032313 (2008).
 - [8] Y.-C. Ou, H. Fan, and S.-M. Fei, Phys. Rev. A **78**, 012311 (2008).
 - [9] E. Jung, D. Park, and J.-W. Son, Phys. Rev. A **80**, 010301 (2009).
 - [10] C. Eltschka, A. Osterloh, and J. Siewert, Phys. Rev. A **80**, 032313 (2009).
 - [11] X.-J. Ren, and W. Jiang, Phys. Rev. A **81**, 024305 (2010).
 - [12] M. F. Cornelio, Phys. Rev. A **87**, 032330 (2013).
 - [13] B. Regula, S. D. Martino, S. Lee, and G. Adesso, Phys. Rev. Lett. **113**, 110501 (2014).
 - [14] J. S. Kim, Phys. Rev. A **90**, 062306 (2014).
 - [15] Y.-K. Bai, M.-Y. Ye, and Z. D. Wang, Phys. Rev. A **80**, 044301 (2009).

- (2009)
- [16] G. H. Aguilar, A. Valdés-Hernández, L. Davidovich, S. P. Walborn, and P. H. Souto Ribeiro, Phys. Rev. Lett. **113**, 240501 (2014).
 - [17] C.-S. Yu and H.-S. Song, Phys. Rev. A **71**, 042331 (2005).
 - [18] D. Yang, Phys. Lett. A **360**, 249 (2006).
 - [19] D. P. Chi, J. W. Choi, K. Jeong, J. S. Kim, T. Kim, and S. Lee, J. Math. Phys. (N.Y.) **49**, 112102 (2008).
 - [20] C.-S. Yu, and H.-S. Song, Phys. Rev. A **77**, 032329 (2008).
 - [21] A. Kay, D. Kaszlikowski, and R. Ramanathan, Phys. Rev. Lett. **103**, 050501 (2009).
 - [22] A. García-Sáez and J. I. Latorre, Phys. Rev. B **87**, 085130 (2013).
 - [23] A. Osterloh and R. Schutzhold, Phys. Rev. B **91**, 125114 (2015).
 - [24] C. Eltschka, and J. Siewert, Phys. Rev. Lett. **114**, 140402 (2015).
 - [25] G. Adesso, and F. Illuminati, New J. Phys. **8**, 15 (2006).
 - [26] T. Hiroshima, G. Adesso, and F. Illuminati, Phys. Rev. Lett. **98**, 050503 (2007).
 - [27] G. Adesso and F. Illuminati, Phys. Rev. Lett. **99**, 150501 (2007).
 - [28] M. Koashi, and A. Winter, Phys. Rev. A **69**, 022309 (2004).
 - [29] M. Christandl, and A. Winter, J. Math. Phys. (N.Y.) **45**, 829 (2004).
 - [30] Dong Yang, K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim, and Wei Song, IEEE. Tran. Info. Theory, **55**, 3375 (2009).
 - [31] Y.-C. Ou and H. Fan, Phys. Rev. A **75**, 062308 (2007).
 - [32] J. S. Kim, A. Das, and B. C. Sanders, Phys. Rev. A **79**, 012329 (2009).
 - [33] H. He, and G. Vidal, Phys. Rev. A **91**, 012339 (2015).
 - [34] J. H. Choi and J. S. Kim, Phys. Rev. A **92**, 042307 (2015).
 - [35] Y. Luo and Y. Li, Ann. Phys. **362**, 511 (2015).
 - [36] J. S. Kim, and B. C. Sanders, J. Phys. A: Math. Theor. **43**, 445305 (2010).
 - [37] M. F. Cornelio and M. C. de Oliveira, Phys. Rev. A **81**, 032332 (2010).
 - [38] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. Lett. **97**, 260502 (2006).
 - [39] Y.-K. Bai, M.-Y. Ye, and Z. D. Wang, Phys. Rev. A **78**, 062325 (2008).
 - [40] Y.-K. Bai, N. Zhang, M.-Y. Ye, and Z. D. Wang, Phys. Rev. A **88**, 012123 (2013).
 - [41] T. R. de Oliveira, M. F. Cornelio, and F. F. Fanchini, Phys. Rev. A **89**, 034303 (2014).
 - [42] Y.-K. Bai, Y.-F. Xu, and Z. D. Wang, Phys. Rev. Lett. **113**, 100503 (2014).
 - [43] X.-N. Zhu and S.-M. Fei, Phys. Rev. A **90**, 024304 (2014).
 - [44] Y.-K. Bai, Y.-F. Xu, and Z. D. Wang, Phys. Rev. A **90**, 062343 (2014).
 - [45] F. Liu, F. Gao, and Q.-Y. Wen, Sci. Rep. **5**, 16745 (2015).
 - [46] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996).
 - [47] J. Cui, M. Gu, L. C. Kwek, M. F. Santos, H. Fan, and V. Vedral, Nature Commun. **3**, 812 (2012).
 - [48] F. Franchini, J. Cui, L. Amico, H. Fan, M. Gu, L. C. Kwek, V. Korepin, and V. Vedral, Phys. Rev. X **4**, 041028 (2014).
 - [49] S. T. Flammia, A. Hamma, T. L. Hughes, and X. G. Wen, Phys. Rev. Lett. **103**, 261601 (2009).
 - [50] G. B. Halasz, and A. Hamma, Phys. Rev. Lett. **110**, 170605 (2013).
 - [51] A. Hamma, L. Cincio, S. Santra, P. Zanardi, and A. Luigi, Phys. Rev. Lett. **110**, 210602 (2013).
 - [52] J. Cui, L. Amico, H. Fan, M. Gu, A. Hamma, and V. Vedral, Phys. Rev. B **88**, 125117 (2013).
 - [53] Yu-Xin Wang, Liang-Zhu Mu, and V. Vedral, and Heng Fan, arXiv:1504.03909.
 - [54] Y.-C. Ou, Phys. Rev. A **75**, 034305 (2007).